Exact real arithmetical algorithms in binary continued fractions

Petr Kůrka
Center for Theoretical Study
Academy of Sciences and Charles University in Prague

22nd IEEE Symposium on Computer Arithmetic
Lyon, june 24, 2015

Exact real computation: arbitrary precision

The first k digits of $x + y$, $xy, \ldots$ depend only on the first n_k digits of x and y

Exact real computation is possible in redundant number systems whose digits represent Möbius transformations.

Möbius transformations

$$M(z) = \frac{az+b}{cz+d}, \ \det(M) = ad - bc \neq 0$$

$M : \overline{\mathbb{R}} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ the extended real line

The space $\mathbb{M}(\mathbb{R})$ of transformations is a group.

$F = \{F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}} : \ a \in A\}$ Möbius iterative system

$$F_u = F_{u_0} F_{u_1} \cdots F_{u_{n-1}}, \ u \in A^n$$

The convergence space

$$\mathbb{X}_F = \{u \in A^\omega : \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}$$

$$u \in \mathbb{X}_F \iff \forall z \in \mathbb{C} \setminus \mathbb{R}, \lim_{n\to\infty} F_{u_{[0,n)}}(z) \in \overline{\mathbb{R}}$$

$\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ the value map

$$\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}},\ u \in \mathbb{X}_F$$

Möbius number system (F, Σ):

$\Sigma \subseteq \mathbb{X}_F$ is a subshift

$\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective.

Subshifts

$D \subseteq A^*$: forbidden words

$$\Sigma_D = \{u \in A^{\mathbb{N}} : \forall v \in D, v \not\sqsubseteq u\}$$

the language of a subshift:

$$\mathcal{L}(\Sigma) = \{v \in A^* : \exists u \in \Sigma, v \sqsubseteq u\}$$

Σ is a sofic subshift if $\mathcal{L}(\Sigma)$ is regular.
$\mathcal{L}(\Sigma)$: labels of paths of a finite labelled graph

Positional systems

$\beta > 1$: base

$A = [p, p+1, \dots, q] \subset \mathbb{Z}$ alphabet

$F_a(z) = \frac{z+a}{\beta}$ transformations

$$F_u(z) = \frac{u_0}{\beta} + \cdots + \frac{u_{n-1}}{\beta^n} + \frac{z}{\beta^n}$$

$$\Phi(u) = \sum_{n=0}^{\infty} \frac{u_n}{\beta^{n+1}}, \quad \Phi : A^\omega \to \left[\frac{p}{\beta - 1}, \frac{q}{\beta - 1}\right]$$

$F_{\overline{0}}(x) = \beta x$: number systems for $\overline{\mathbb{R}}$

$$\Phi(\overline{0}^n u) = \sum_{k=0}^{\infty} u_k \beta^{n-k-1}$$

Simple continued fractions $A = \{0, 1\}$

$$F_1(z) = z + 1,\ F_0(z) = \frac{z}{z+1} = \frac{1}{1+\frac{1}{z}}$$

$$F_1^a(z) = a + z,\ F_0^a(z) = \frac{1}{a + \frac{1}{z}}$$

$u = 1^{a_0} 0^{a_1} 1^{a_2} \cdots 1^{a_{2n}}$, $a_i > 0$ for $i > 0$

$$\begin{aligned} F_u(x) &= a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_{2n} + x}} \\ &= a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +} \cdots + \frac{1}{a_{2n} + x} \end{aligned}$$

Simple continued fractions

$$u = 1^{a_0}0^{a_1}1^{a_2}\cdots 0^{a_{2n+1}},\ a_i > 0 \text{ for } i > 0$$

$$\begin{aligned} F_u(x) &= a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_{2n+1} + \frac{1}{x}}} \\ &= a_0 + \frac{1}{a_1 +} \frac{1}{a_2 + \cdots +} \frac{1}{a_{2n+1} +} \frac{1}{x} \end{aligned}$$

$$u = 1^{a_0}0^{a_1}1^{a_2}0^{a_3}\cdots,\ a_i > 0 \text{ for } i > 0$$

$$\Phi(u) = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 + \cdots},$$

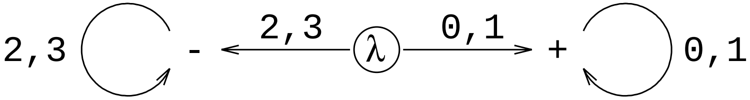
$$\Phi : \{0,1\}^\omega \to [0,\infty]$$

Symmetric continued fractions $A = \{0, 1, 2, 3\}$

$\Sigma = \{0, 1\}^\omega \cup \{2, 3\}^\omega$

$$F_0(x) = \frac{x}{x+1}, F_1(x) = x + 1,$$
$$F_2(x) = \frac{x}{1-x}, F_3(x) = x - 1$$

$$\begin{aligned}
V_+ &= [0, \infty] = F_0(V_+) \cup F_1(V_+) = [0, 1] \cup [1, \infty] \\
V_- &= [\infty, 0] = F_3(V_-) \cup F_2(V_-) = [\infty, -1] \cup [-1, 0] \\
V_\lambda &= \overline{\mathbb{R}} = F_0(V_+) \cup F_1(V_+) \cup F_2(V_-) \cup F_3(V_-)
\end{aligned}$$



Sofic number system (F, G, V) over A

$F = \{F_a : \ a \in A\}$: iterative system
$G = (B, E, \mathbf{i})$, $E \subseteq B \times A \times B$: labelled graph
Σ_G: labels of paths in G

$$(p, a, q) \in E \iff p \xrightarrow{a} q$$

$V = \{V_p : \ p \in B\}$ closed intervals

$$V_p = \bigcup \{F_a(V_q) : \ p \xrightarrow{a} q\}, \ V_{\mathbf{i}} = \overline{\mathbb{R}}$$

If $p \xrightarrow{a} q$ then F_a is contractive on V_q

The unary graph

$$(X, p, q) \in \mathbb{M}(\mathbb{R}) \times B \times B : \; X(V_p) \subseteq V_q$$

vertices

$$(X, p, q) \xrightarrow{a,\lambda} (XF_a, r, q) \text{ if } p \xrightarrow{a} r$$

absorption edge

$$(X, p, q) \xrightarrow{\lambda,a} (F_a^{-1}X, p, r) \text{ if } q \xrightarrow{a} r,\; X(V_p) \subseteq F_a(V_r)$$

emission edge

If $(X, \mathbf{i}, \mathbf{i}) \xrightarrow{u,v}$ then $\Phi(v) = X(\Phi(u))$.

Modular systems: $\det(F_a) = 1$

Theorem[Raney] In symmetric continued fractions, the unary algorithm can be computed by a finite state transducer.

Theorem[Delacourt and Kůrka]
In modular systems the norm of the state matrix of the unary algorithm remains bounded.

Theorem[Kůrka and Vávra] Möbius transformations are the only analytic functions computable by finite state transducers.

Fractional bilinear functions

$$P(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h}, \; M(x) = \frac{ax + b}{cx + d}$$

$(P^* x)(y) = P(x,y)$, $(P_* y)(x) = P(x,y)$

$P^* x$, $P_* y$ are transformations.

$(P^* M)(x,y) = P(M(x), y)$,

$(P_* M)(x,y) = P(x, M(y))$,

$(MP)(x,y) = M(P(x,y))$

$P^* M, P_* M, MP$ are fractional bilinear functions.

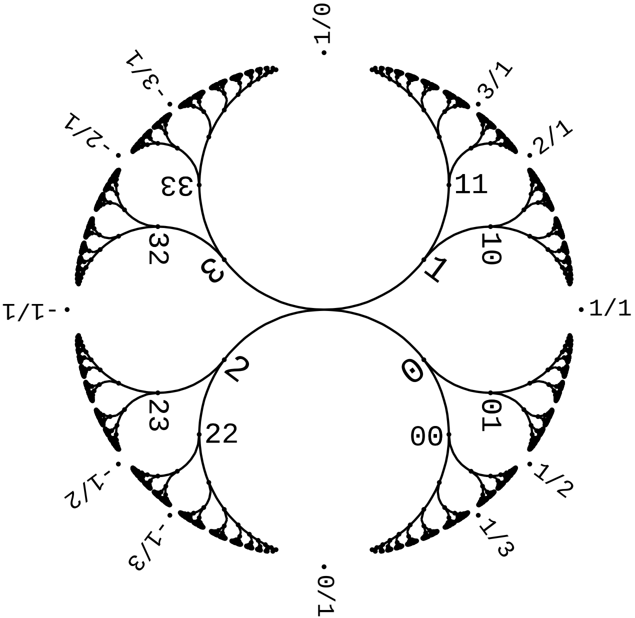
The bilinear graph

(X, p, q, r) : $X(V_p, V_q) \subseteq V_r$: vertices

$$(X, p, q, r) \xrightarrow{a,\lambda,\lambda} (X^* F_a, p', q, r) \text{ if } p \xrightarrow{a} p'$$
$$(X, p, q, r) \xrightarrow{\lambda,a,\lambda} (X_* F_a, p, q', r) \text{ if } q \xrightarrow{a} q'$$
$$(X, p, q, r) \xrightarrow{\lambda,\lambda,a} (F_a^{-1} X^*, p, q, r') \text{ if } r \xrightarrow{a} r',$$
$$X(V_p, V_q) \subseteq F_a(V_r')$$

If $(X, \mathbf{i}, \mathbf{i}, \mathbf{i}) \xrightarrow{u,v,w}$ then $\Phi(w) = X(\Phi(u), \Phi(v))$.

Modular systems are not redundant and converge slowly



The binary and prefix codes

$\mathbf{b} : \mathbb{N} \to \{0,1\}^+$: the binary code

$$\mathbf{p} : \{1, 2, \ldots, \infty\} \to \{0,1\}^+ \cup \{1^\omega\}$$

$$n = 2^k + 2^{k-1}u_0 + \cdots u_{k-1}$$

$$\mathbf{p}(n) = 1^k 0 u, \ |\mathbf{p}(n)| = 2\lceil \ln n \rceil$$

n	$\mathbf{b}(n)$	$\mathbf{p}(n)$
0	0	–
1	1	0
2	10	100
3	11	101
4	100	11000

n	$\mathbf{b}(n)$	$\mathbf{p}(n)$
5	101	11001
6	110	11010
7	111	11011
8	1000	1110000
9	1001	1110001
∞	–	1^ω

The compression code is continuous and one-to-one

$$\mathbf{c} : \{0,1\}^\omega \cup \{2,3\}^\omega \to \{0,1\}^\omega$$

$$\begin{aligned}
\mathbf{c}(0^{a_0}1^{a_1}0^{a_2}\cdots) &= 00\mathbf{p}(a_0)\mathbf{p}(a_1)\mathbf{p}(a_2)\cdots \\
\mathbf{c}(1^{a_0}0^{a_1}1^{a_2}\cdots) &= 01\mathbf{p}(a_0)\mathbf{p}(a_1)\mathbf{p}(a_2)\cdots \\
\mathbf{c}(2^{a_0}3^{a_1}2^{a_2}\cdots) &= 10\mathbf{p}(a_0)\mathbf{p}(a_1)\mathbf{p}(a_2)\cdots \\
\mathbf{c}(3^{a_0}2^{a_1}3^{a_2}\cdots) &= 11\mathbf{p}(a_0)\mathbf{p}(a_1)\mathbf{p}(a_2)\cdots
\end{aligned}$$

$$\forall i, a_i > 0$$

Binary continued fractions

$\Phi : \{0,1\}^\omega \cup \{2,3\}^\omega \to \overline{\mathbb{R}}$: Symmetric CF

$$\Psi = \Phi \circ \mathbf{c} : \{0,1\}^\omega \to \overline{\mathbb{R}} : \quad \text{BCF}$$

The unary and binary algorithms use the compression and decompression transducers with states $(s, a, b, n) \in \{0,1\}^3 \times \mathbb{N}$,
$u = 0^{a_0} 1^{a_1} \cdots 0^{a_k} \implies n = \lceil \ln a_k \rceil$

State matrices of the unary algorithm are bounded.
State matrices of the binary algorithm grow slowly.

Binary continued fractions