Faster multiprecision integer division

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  - Quotient Q and remainder R

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- Require v_0 *normalised*
- If $B = 2^{32}$ or 2^{64} then $B < v_0 \leq B/2$
- Also require $u_1 < v_0$

Schoolbook algorithm

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  - $q_i \leftarrow$ quotient of $< a_{n+i}, a_{n+i-1} >$ by $< d_{n-1} >$
  - $A \leftarrow A - q_i D B^i$
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- Shift remainder right

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- Need tests and adjustments for both cases

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  - 2000 BC, Babylonian “Clay tablet” PC’s

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Additional problem:

- Quotient limb may now be too small or too large!!

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- One very unlikely correction

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- Möller-Granlund give 3×2 version of their algorithm

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- Multiprecision corrections rare, always in same direction
- Moreover, MP corrections in *same* direction as 3×2 corrections

Timings (MP divappr)

limbs	*GMP 6.0.0a*	*Our code*
3	$6.7e-8s$	$5.7e-8s$
4	$8.8e-8s$	$7.5e-8s$
6	$1.36e-7s$	$1.17e-7s$
7	$1.85e-7s$	$1.41e-7s$
9	$2.27e-7s$	$1.89e-7s$
11	$2.98e-7s$	$2.38e-7s$
15	$4.35e-7s$	$3.64e-7s$
19	$6.07e-7s$	$4.89e-7s$
21	$7.05e-7s$	$5.96e-7s$
27	$1.05e-6s$	$8.7e-7s$
33	$1.45e-6s$	$1.2e-6s$
37	$1.72e-6s$	$1.48e-6s$

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- One correction with unpredicted branch, one unlikely correction
- One additional 2 limb subtraction over Möller-Granlund
- No real time difference to Möller-Granlund for MP division

Outline

Improvement 3: divapprox

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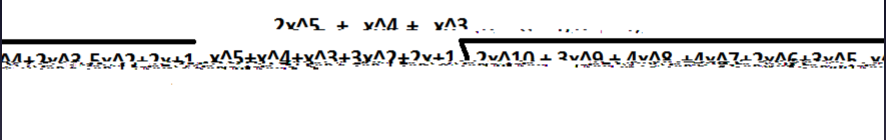
- Complexity same as schoolbook divrem
- Can we compute integer divapprox in subquadratic time

Middle product

2004, Hanrot, Quercia, Zimmermann



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$$
\begin{array}{r|l}
 & 2x^5 + x^4 + x^3 \\
\hline
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- $\mathrm{Mulmid}_{m,n}(A, B) = < ov_1, ov_0, p_{m-n}, \ldots, p_1, p_0 > = \sum_{i+j=n-1}^{m-1} a_i b_j B^{i+j-n+1}$

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- Roughly : computes middle third of $2n \times n$ product

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- – In total $(sh + 2) + sl = s + 2$ limbs of A affected
- Recursively compute low sl limbs of $Q^\star$ using DC divapprox

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- Due to massive cancellation of correction terms, there is a linear time fixup to yield correct base B arithmetic

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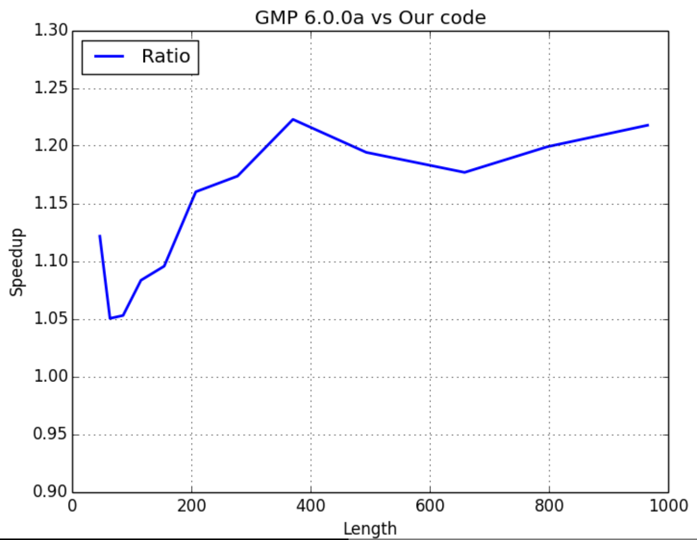

Thank You

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